

GALOIS GROUPS AND COHOMOLOGICAL FUNCTORS

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ABSTRACT. Let $q = p^s$ be a prime power, F a field containing a root of unity of order q , and G_F its absolute Galois group. We determine a new canonical quotient $\text{Gal}(F_{(3)}/F)$ of G_F which encodes the full mod- q cohomology ring $H^*(G_F, \mathbb{Z}/q)$ and is minimal with respect to this property. We prove some fundamental structure theorems related to these quotients. In particular, it is shown that when $q = p$ is an odd prime, $F_{(3)}$ is the compositum of all Galois extensions E of F such that $\text{Gal}(E/F)$ is isomorphic to $\{1\}$, \mathbb{Z}/p or to the nonabelian group H_{p^3} of order p^3 and exponent p .

1. INTRODUCTION

Let p be a fixed prime number and $q = p^s$ a fixed p -power, where $s \geq 1$. For a profinite group G , let $H^*(G) = \bigoplus_{i=0}^{\infty} H^i(G, \mathbb{Z}/q)$ be the cohomology (graded) ring with the trivial action of G on \mathbb{Z}/q . We will be mostly interested in the case where $G = G_F$ is the absolute Galois group of a field F which contains a root of unity of order q . The ring $H^*(G_F)$, and even its degree ≤ 2 part, is known to encode important arithmetical information on F (see below). In this paper we ask how much Galois-theoretic information is needed to compute $H^*(G_F)$. More specifically, we characterize the minimal quotient of G_F which determines it. In [CEM12] it was shown that $H^*(G_F)$ is determined by a quite small quotient $G_F^{[3]} = G_F/(G_F)^{(3)}$ of G_F . Here $G^{(i)} = G^{(i,q)}$, $i = 1, 2, 3, \dots$, is the *descending q -central sequence* of a given profinite G (see §2). Namely, the inflation map $\text{inf}: H^*(G_F^{[3]})_{\text{dec}} \rightarrow H^*(G_F)$ is an isomorphism, where $H^*(G_F^{[3]})_{\text{dec}}$ denotes the subring of $H^*(G_F^{[3]})$ generated by degree 1 elements. In the present paper we first find an even smaller quotient $(G_F)_{[3]} = G_F/(G_F)_{(3)}$ of G_F with the same property. Here for a profinite group G we set $p > 2$ by $G_{(3)} = G^q[G^{(2)}, G]$ if $p > 2$, and $G_{(3)} = G^{2q}[G^{(2)}, G]$ if $p = 2$ (see §2). Thus, when $q = p$, $G_{(3)} = G^{(3)}$ is

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the third term in the Zassenhaus p -filtration of G . Moreover, we prove that this quotient is *minimal* with respect to this property (see Example 5.2(1)):

Theorem A. *Let N be a closed normal subgroup of G_F . Then the inflation map $H^*(G_F/N)_{\text{dec}} \rightarrow H^*(G_F)$ is an isomorphism if and only if $N \leq (G_F)_{(3)}$. In particular, $(G_F)_{[3]}$ determines $H^*(G_F)$.*

Conversely, we prove (see Corollary 7.3(1)):

Theorem B. *The degree ≤ 2 part of $H^*(G_F)$ determines $(G_F)_{[3]}$.*

We also prove the following “anabelian” result, which strengthens [CEM12, Th. C, D]. Denote the maximal pro- p Galois group of F by $G_F(p)$.

Theorem C. *Let F_1, F_2 be fields containing a q th root of unity. Let $\pi: G_{F_1}(p) \rightarrow G_{F_2}(p)$ be a continuous homomorphism and $\pi^*: H^*(G_{F_2}(p)) \rightarrow H^*(G_{F_1}(p))$, $\pi_{[3]}: (G_{F_1})_{[3]} \rightarrow (G_{F_2})_{[3]}$ the induced maps. Then π is an isomorphism if and only if $\pi_{[3]}$ is an isomorphism if and only if π^* is an isomorphism.*

See Corollary 6.2 and Proposition 5.4.

Regarding the structure of the quotient $(G_F)_{[3]}$, we prove that it is the Galois group of the compositum of all Galois extensions of F with certain specific Galois groups:

Theorem D. *Assume that $q = p \neq 2$ is prime and let F be as above. Then $(G_F)_{(3)}$ is the intersection of all normal open subgroups N of G_F such that G_F/N is isomorphic to \mathbb{Z}/p or H_{p^3} .*

Here H_{p^3} is the non-abelian group of odd order p^3 and exponent p (the Heisenberg group):

$$H_{p^3} = \langle r, s, t \mid r^p = s^p = t^p = [r, t] = [s, t] = 1, [r, s] = t \rangle.$$

In the remaining case $q = p = 2$ the group $(G_F)_{(3)}$ is known to be the intersection of all normal open subgroups N of G_F such that G_F/N is isomorphic to either $\mathbb{Z}/2$, $\mathbb{Z}/4$, or the dihedral group D_4 of order 8 (Villegas [Vil88], Mináč–Spira [MSp96, Cor. 2.18]; see also [EM11, Cor. 11.3 and Prop. 3.2]). Moreover, $\mathbb{Z}/2$ can be omitted from this list unless F is Euclidean [EM11, Cor. 11.4].

Theorem A gives a new restriction on the structure of maximal pro- p Galois groups of fields as above. Indeed, it implies that if the defining relations in such a group $G = G_F(p)$ are changed within $G_{(3)}$, then the resulting group cannot be realized as a Galois group in this way (see Corollary

6.3 for a precise statement). In particular, Theorem A directly implies the classical Artin–Schreier theorem, asserting that elements of absolute Galois groups can have only order 1, 2, or ∞ (Remark 6.4(2)).

Our proofs of Theorems A, C, D are based on the bijectivity of the Galois symbol homomorphism $K_*^M(F)/qK_*^M(F) \rightarrow H^*(G_F)$, proved by Rost and Voevodsky (with a patch by Weibel), where $K_*^M(F)$ is the Milnor K -ring of F ; see [Voe03], [Voe11], [Wei09], [Wei08]. The bijectivity of the Galois symbol in degree 2 was proved earlier by Merkurjev and Suslin [MS82]. Specifically, the proofs of Theorems A and the second equivalence in Theorem C use the bijectivity of this map, the proof of the first equivalence in Theorem C uses its bijectivity in degree 2, and that of Theorem D uses only its injectivity in degree 2.

Our approach is purely group-theoretic, and is based on a fundamental notion of duality between a pair (T, T_0) of normal subgroups of a profinite group G and a subgroup A of the second cohomology $H^2(G/T)$ (see §3). When applied to $T = G^{(2)}$, $T_0 = G_{(3)}$ and $A = H^2(G^{[2]})_{\text{dec}}$, with $G = G_F$ as above, it leads to Theorems A–D. Moreover, it can also be applied to other choices of T, T_0, A as well to yield analogous results. Notably, taking $T = G^{(2)}$, $T_0 = G^{(3)}$ and $A = H^2(G^{[2]})$, we strengthen the main results of [CEM12]. As a third example we may take $T = G^{(2)}$, $T_0 = G^{q^2}[G, G]$ and $A = \text{Im}(\beta_{G^{[2]}})$, where $\beta_{G^{[2]}}: H^1(G^{[2]}) \rightarrow H^2(G^{[2]})$ is the Bockstein map (see §2).

The Galois group $(G_F)_{[3]}$ encodes important arithmetical information on F . For instance, when $q = 2$ it was shown in [MSp90], [MSp96] that $(G_F)_{[3]}$ and the Kummer element of -1 encode the orderings on F as well as the Witt ring of quadratic forms over F . Moreover, in [EM12] we use Theorem D to prove that $(G_F)_{[3]}$ also encodes a class of valuations which are important in the pro- p context, namely, (Krull) valuations v whose value group is not divisible by p and whose 1-units are p -powers (this is a weak form of Hensel’s lemma). Specifically, under a finiteness assumption, and assuming the existence of a root of unity of order 4 in F when $p = 2$, there exists such a valuation if and only if the center $Z((G_F)_{[3]})$ has a nontrivial image in $G_F^{[2]}$.

For interesting connections between this kind of results and birational anabelian geometry see [BT12].

2. PRELIMINARIES

A) Profinite groups. We work in the category of profinite groups. Thus all homomorphisms of profinite groups will be tacitly assumed to be continuous and all subgroups will be closed. The **descending q -central filtration** $G^{(i)} = G^{(i,q)}$ of a profinite group G is defined inductively by

$$G^{(1)} = G, \quad G^{(i)} = (G^{(i-1)})^q [G^{(i-1)}, G], \quad i \geq 2.$$

Thus $G^{(i)}$ is generated by all q th powers of elements of $G^{(i-1)}$ and all commutators $[h, g] = h^{-1}g^{-1}hg$, where $h \in G^{(i-1)}$, $g \in G$. Set $G^{[i]} = G/G^{(i)}$. We also define

$$(2.1) \quad \delta = \begin{cases} 1 & p > 2 \\ 2 & p = 2, \end{cases} \quad G_{(3)} = G^{\delta q} [G^{(2)}, G], \quad G_{[3]} = G/G_{(3)}.$$

2.1. Remarks. (1) As $[h, g] = h^{-2}(hg^{-1})^2g^2$, we have $[G, G] \leq G^2$. Therefore, when $q = 2$ we have $G^{(2)} = G^2$, so $G^{(3)} = G^4[G^{(2)}, G] = G_{(3)}$.

(2) One has $G^{(3)} \leq G_{(3)}$. Indeed, this is immediate when $p > 2$. When $p = 2$ we have $g^{q^2} = (g^{q/2})^{2q}$ for every $g \in G$, so $(G^q)^q \leq G^{2q}[G^q, G]$. Further, one has the identities (see [Lab66, Prop. 5] and its proof)

$$(g_1g_2)^q \equiv g_1^qg_2^q[g_2, g_1]^{\binom{q}{2}}, \quad [g_1g_2, h] \equiv [g_1, h][g_2, h] \pmod{[[G, G], G]}$$

It follows that $[g_1^2, g_2^2]^{\binom{q}{2}} \in G^{2q}[[G, G], G]$ for any $g_1, g_2 \in G$. By (1), $[G, G]^q \leq (G^2)^q \leq G^{2q}[[G, G], G]$. Consequently, $(G^{(2)})^q \leq G^{2q}[G^{(2)}, G] = G_{(3)}$, whence our claim.

B) Profinite cohomology. We refer to [NSW08], [RZ10], or [Ser02] for basic notions and facts in profinite cohomology. In particular, given a profinite group G , let $H^i(G) = H^i(G, \mathbb{Z}/q)$ be the i th profinite cohomology group of G with respect to its trivial action on \mathbb{Z}/q . Thus $H^1(G) = \text{Hom}_{\text{cont}}(G, \mathbb{Z}/q)$. Let $H^*(G) = \bigoplus_{r=0}^{\infty} H^r(G)$ be the cohomology ring with the cup product \cup . Given a homomorphism $\pi: G_1 \rightarrow G_2$ of profinite groups, let $\pi_r^*: H^r(G_2) \rightarrow H^r(G_1)$ and $\pi^*: H^*(G_2) \rightarrow H^*(G_1)$ be the induced homomorphisms. We write $\text{res}, \text{inf}, \text{trg}$ for the restriction, inflation, and transgression maps, respectively. For a normal subgroup N of G , there is a canonical action of G on $H^i(N)$. When $i = 1$ it is given by $\psi \mapsto \psi^g$, where $\psi^g(n) = \psi(g^{-1}ng)$ for $g \in G$ and $n \in N$. We denote the group of all G -invariant elements of $H^i(N)$ by $H^i(N)^G$.

For each $r \geq 0$, the cup product induces a homomorphism $H^1(G)^{\otimes r} \rightarrow H^r(G)$ of \mathbb{Z}/q -modules. Let $H^r(G)_{\text{dec}}$ be its image (where $H^0(G)_{\text{dec}} =$

\mathbb{Z}/q), and let $H^*(G)_{\text{dec}} = \bigoplus_{r=0}^{\infty} H^r(G)_{\text{dec}}$ be the **decomposable cohomology ring** with the cup product.

Following [CEM12, §3], set $\widehat{H^*(G)} = \bigoplus_{r=0}^{\infty} H^1(G)^{\otimes r} / C_r$, where C_r is the subgroup of $H^1(G)^{\otimes r}$ generated by all elements $\psi_1 \otimes \cdots \otimes \psi_r$ such that $\psi_i \cup \psi_j = 0$ for some $i < j$. It is a graded ring and there is a canonical graded ring epimorphism $\omega_G: \widehat{H^*(G)} \rightarrow H^*(G)_{\text{dec}}$. The ring $\widehat{H^*(G)}$ and the map ω_G are functorial in the natural sense. We call $H^*(G)$ **quadratic** (resp., **r -quadratic**) if ω_G is an isomorphism (resp., in degree r).

Lemma 2.2. *When $H^*(G)$ is 2-quadratic, the kernel of $\inf_G: H^2(G^{[2]})_{\text{dec}} \rightarrow H^2(G)$ is generated by kernel elements of the form $\psi \cup \psi'$, with $\psi, \psi' \in H^1(G^{[2]})$.*

Proof. This follows from the commutative diagram

$$\begin{array}{ccccc} H^1(G^{[2]})^{\otimes 2} & \xrightarrow[\sim]{\inf_G} & H^1(G)^{\otimes 2} & \longrightarrow & H^1(G)^{\otimes 2} / C_2 \\ \cup \downarrow & & & & \downarrow \cup \\ H^2(G^{[2]})_{\text{dec}} & \xrightarrow{\inf} & H^2(G)_{\text{dec}} & & \end{array}$$

and the definition of C_2 . □

The **Bockstein map** $\beta_G: H^1(G) \rightarrow H^2(G)$ is the connecting homomorphism arising from the short exact sequence of trivial G -modules

$$0 \rightarrow \mathbb{Z}/q \rightarrow \mathbb{Z}/q^2 \rightarrow \mathbb{Z}/q \rightarrow 0.$$

It is functorial in the natural way. The following lemma was proved in [EM11, Cor. 2.11] for I finite, and follows in general by a limit argument. See also Proposition 10.3 for an alternative proof.

Lemma 2.3. *If $G = (\mathbb{Z}/q)^I$ for some I , then $H^2(G) = \text{Im}(\beta_G) + H^2(G)_{\text{dec}}$.*

For $r \geq 0$ let B_r be the subgroup of $H^1(G)^{\otimes r}$ generated by all elements $\psi_1 \otimes \cdots \otimes \psi_r$ such that $\beta_G(\psi_i) = 0$ for some i . We define a graded ring $H^*(G)_{\text{Bock}} = \bigoplus_{r=0}^{\infty} H^1(G)^{\otimes r} / B_r$ with the tensor product.

C) Galois cohomology. The following theorem collects the cohomological properties of an absolute Galois group which are needed for this paper.

Theorem 2.4. *Let F be a field containing a root of unity ζ_q of order q and let G_F be its absolute Galois group. Then:*

- (i) $G_F^{[2]} \cong (\mathbb{Z}/q)^I$;
- (ii) *there exists $\xi \in H^1(G_F)$ with $\beta_G(\psi) = \psi \cup \xi$ for every $\psi \in H^1(G_F)$;*
- (iii) $H^*(G_F) = H^*(G_F)_{\text{dec}}$;

(iv) $H^*(G_F)$ is quadratic.

Proof. We identify the group μ_p of q th roots of unity in F with \mathbb{Z}/q , where ζ_q corresponds to 1 mod q . The obvious map $F^\times/(F^\times)^q \rightarrow F^\times/(F^\times)^p$ is surjective, and by the Kummer isomorphism, so is the functorial map $H^1(G_F) = H^1(G_F, \mathbb{Z}/q) \rightarrow H^1(G_F, \mathbb{Z}/p)$. This implies (i). In (ii) one takes ξ to be the Kummer element corresponding to ζ_q (see [EM11, Prop. 3.2]). Condition (iii) (resp., (iv)) follows from the surjectivity (resp., injectivity) of the Galois symbol $K_*^M(F)/q \rightarrow H^*(G_F)$, as proved by Rost, Voevodsky, and Weibel (see [CEM12, §8]). \square

Remark 2.5. Many of our results do not require the full strength of conditions (iii) and (iv) of Theorem 2.4, but only their validity in degrees 1 and 2. These weaker facts follow from the Kummer isomorphism and the Merkurjev–Suslin theorem ([MS82], [GS06]), respectively.

Remark 2.6. By [CEM12, Remark 8.2], $\inf: H^*(G_F(p)) \rightarrow H^*(G_F)$ is an isomorphism. Therefore (i)–(iv) hold also for $G_F(p)$.

3. DUALITY

Let G be a profinite group and let T, T_0 be normal subgroups of G such that $T^q[T, G] \leq T_0 \leq T \leq G^{(2)}$. Let

$$K = \text{Ker}(H^1(T)^G \xrightarrow{\text{res}} H^1(T_0)), \quad K' = \text{Ker}(H^2(G/T) \xrightarrow{\inf} H^2(G/T_0)).$$

Note that if $T_0 = T^q[T, G]$, then $K = H^1(T)^G$. As $T, T_0 \leq G^{(2)}$, the inflations $H^1(G/T) \rightarrow H^1(G)$, $H^1(G/T_0) \rightarrow H^1(G)$ are isomorphisms. The functoriality of the 5-term sequence [NSW08, pp. 78–79] gives a commutative diagram with exact rows

$$(3.1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & H^1(T)^G & \xrightarrow{\text{trg}} & H^2(G/T) & \xrightarrow{\inf} & H^2(G) \\ & & \downarrow \text{res} & & \downarrow \inf & & \parallel \\ 0 & \longrightarrow & H^1(T_0)^G & \xrightarrow{\text{trg}} & H^2(G/T_0) & \xrightarrow{\inf} & H^2(G). \end{array}$$

By the snake lemma, K, K' are therefore isomorphic via transgression.

There is a perfect duality

$$(3.2) \quad T/T^q[T, G] \times H^1(T)^G \rightarrow \mathbb{Z}/q, \quad (\bar{\sigma}, \psi) \mapsto \psi(\sigma)$$

which is functorial in the natural sense [EM11, Cor. 2.2]. It induces a (functorial) perfect duality

$$(3.3) \quad T/T^q[T, G] \times \text{trg}(H^1(T)^G) \rightarrow \mathbb{Z}/q, \quad \langle \bar{\sigma}, \varphi \rangle \mapsto (\text{trg}^{-1}(\varphi))(\sigma).$$

Proposition 3.1. (3.2) and (3.3) induce perfect pairings

- (a) $(\cdot, \cdot): T/T_0 \times K \rightarrow \mathbb{Z}/q, (\sigma T_0, \psi) \mapsto \psi(\sigma);$
- (b) $\langle \cdot, \cdot \rangle: T/T_0 \times K' \rightarrow \mathbb{Z}/q, \langle \sigma T_0, \varphi \rangle = (\text{trg}^{-1}(\varphi))(\sigma).$

Proof. (a) A perfect pairing of abelian groups $(\cdot, \cdot): A \times B \rightarrow C$ induces for any $A_0 \leq A$ a perfect pairing $(A/A_0) \times \{b \in B \mid (A_0, b) = 0\} \rightarrow C$. Take $A = T/T^q[T, G]$, $B = H^1(T)^G$, $C = \mathbb{Z}/q$, and let A_0 be the image of T_0 in A . By (3.2), the substitution pairing $A \times B \rightarrow \mathbb{Z}/q$ is perfect. Furthermore, for $\psi \in H^1(T)^G$ we have $(A_0, \psi) = 0$ if and only if $\psi \in K$.

(b) follows from (a). \square

For the connection between the second cohomology and central extensions see e.g., [NSW08, Th. 1.2.4] or [RZ10, §6.8].

Proposition 3.2. Let A be a subgroup of $H^2(G/T)$ and let A_0 be a set of generators of $A \cap \text{trg}(H^1(T)^G)$. The following conditions are equivalent:

- (a) $K = \text{trg}^{-1}[A];$
- (b) there is an exact sequence

$$0 \rightarrow K \xrightarrow{\text{trg}} A \xrightarrow{\text{inf}} H^2(G);$$

- (c) there is an exact sequence

$$0 \rightarrow K' \hookrightarrow A \xrightarrow{\text{inf}} H^2(G);$$

- (d) $\langle \cdot, \cdot \rangle$ induces a perfect pairing

$$T/T_0 \times \text{Ker}(A \xrightarrow{\text{inf}} H^2(G)) \rightarrow \mathbb{Z}/q;$$

- (e) T_0 is the annihilator of $A \cap \text{trg}(H^1(T)^G)$ under $\langle \cdot, \cdot \rangle;$
- (f) $T_0 = \bigcap \text{Ker}(\Psi)$, where Ψ ranges over all homomorphisms making a commutative diagram

$$\omega: \quad 0 \longrightarrow \mathbb{Z}/q \longrightarrow C \xrightarrow{\quad} G/T \longrightarrow 1,$$

$\begin{array}{ccc} & & G \\ & \swarrow \Psi & \downarrow \\ & C & G/T \end{array}$

where ω is the central extension corresponding to some $\varphi \in A_0$ (where an empty intersection is interpreted as T).

Proof. The 5-term sequence gives (a) \Leftrightarrow (b) and (d) \Leftrightarrow (e).

The equivalence (b) \Leftrightarrow (c) follows from the bijectivity of $\text{trg}: K \rightarrow K'$.

For (c) \Leftrightarrow (d) use Proposition 3.1(b).

For (e) \Leftrightarrow (f) let $\varphi \in A_0$ and let ω be the corresponding central extension as above. By [Hoe68, 2.1], for every $\psi \in H^1(T)^G$ with $\text{trg}(\psi) = \varphi$ there

is a homomorphism $\Psi: G \rightarrow C$ such that $\psi = \Psi|_T$ and the diagram in (f) is commutative. For such Ψ, ψ and for $\sigma \in T$ and $\varphi \in A_0$ we have $\langle \sigma T_0, \varphi \rangle = \psi(\sigma) = \Psi(\sigma)$. Moreover, $\text{Ker}(\Psi) \leq T$. We conclude that the annihilator of $A \cap \text{trg}(H^1(T)^G)$ in T under $\langle \cdot, \cdot \rangle$ is $\bigcap \text{Ker}(\Psi)$. \square

When these conditions are satisfied, we say that A is **dual** to (T, T_0) .

Examples 3.3. (1) In §10 we will show that, when $G^{[2]} \cong (\mathbb{Z}/q)^I$ for some I , $H^2(G^{[2]})_{\text{dec}}$ is dual to $(G^{(2)}, G_{(3)})$.

(2) For $i \geq 2$, $H^2(G^{[i]})$ is dual to $(G^{(i)}, G^{(i+1)})$. Indeed, here $K = H^1(G^{(i)})^G$ [CEM12, Lemma 5.4], so by the 5-term sequence, (b) holds.

(3) For $\bar{G} = \mathbb{Z}/q$ and for an isomorphism $\psi \in H^1(\bar{G})$, the central extension corresponding to $\beta_{\mathbb{Z}/q}(\psi)$ is

$$0 \rightarrow \mathbb{Z}/q \rightarrow \mathbb{Z}/q^2 \rightarrow \mathbb{Z}/q \rightarrow 0$$

(see [EM11, Prop. 9.2]). For $\bar{G} = (\mathbb{Z}/q)^I$, and for the projection $\psi: \bar{G} \rightarrow \mathbb{Z}/q$ on the i_0 th coordinate, the extension corresponding to $\beta_{\bar{G}}(\psi)$ is then

$$0 \rightarrow \mathbb{Z}/q \rightarrow \prod_{i \in I} C_i \rightarrow \bar{G} \rightarrow 1,$$

where $C_{i_0} = \mathbb{Z}/q^2$ and $C_i = \mathbb{Z}/q$ for $i \neq i_0$, with the natural maps [EM11, Lemma 6.2].

Now assume that G is a profinite group with $G^{[2]} \cong (\mathbb{Z}/q)^I$ for some I . Take $T = G^{(2)}$ and $A = A_0 = \text{Im}(\beta_{G^{[2]}})$. Note that then every homomorphism Ψ as in (e) is necessarily surjective, and $\text{Ker}(\Psi) \leq T = G^{(2)}$. We deduce that the intersection in (f) is

$$G^{(2)} \cap \bigcap \{M \trianglelefteq G \mid G/M \cong \mathbb{Z}/q^2\} = G^{q^2}[G, G].$$

Thus A is dual to $(G^{(2)}, G^{q^2}[G, G])$.

Proposition 3.4. *Suppose that A is dual to (T, T_0) . Then the kernels of $\inf_{G/T_0}: A \rightarrow H^2(G/T_0)$, $\inf_G: A \rightarrow H^2(G)$ coincide.*

Proof. This is straightforward from condition (c) of Proposition 3.2. \square

4. COHOMOLOGICAL DUALITY TRIPLES

In this section we axiomatize several important cases of functorial subgroups of profinite groups, so that we can treat them all in a unified way. For a profinite group G we set $H^{\otimes*}(G) = \bigoplus_{r=0}^{\infty} H^1(G)^{\otimes r}$, considered as an *abelian group*. Assume that we are given:

- (i) subfunctors T, T_0 of the identity functor on the category of profinite groups;
- (ii) a natural transformation α from the functor $G \mapsto H^{\otimes*}(G)$ to the functor $G \mapsto H^2(G)$ (both from the category of profinite groups to the category of abelian groups).

In other words, for every profinite group G we are given subgroups $T(G), T_0(G)$ of G and a group homomorphism $\alpha_G: H^{\otimes*}(G) \rightarrow H^2(G)$, and for every homomorphism $\pi: G_1 \rightarrow G_2$ of profinite groups there are commutative squares

$$\begin{array}{ccccc}
 T(G_1) \hookrightarrow G_1 & T_0(G_1) \hookrightarrow G_1 & H^{\otimes*}(G_2) \xrightarrow{\alpha_{G_2}} H^2(G_2) \\
 \downarrow T(\pi) & \downarrow T_0(\pi) & \downarrow \pi^* \\
 T(G_2) \hookrightarrow G_2 & T_0(G_2) \hookrightarrow G_2 & H^{\otimes*}(G_1) \xrightarrow{\alpha_{G_1}} H^2(G_1)
 \end{array}$$

We denote

$$K(G) = \text{Ker}(\text{res}: H^1(T(G))^G \rightarrow H^1(T_0(G))), \quad A(G) = \text{Im}(\alpha_G).$$

Observe that, in the previous setup, $\pi_2^*(A(G_2)) \subseteq A(G_1)$.

A **cover** of a profinite group G (relative to T) will be an epimorphism $\pi: S \rightarrow G$, where S is a profinite group such that $H^2(S) = 0$ and the induced map $S/T(S) \rightarrow G/T(G)$ is an isomorphism.

Example 4.1. For a profinite group G let $T(G) = G^{(2)}$. The existence of a cover $S \rightarrow G$ means that $G^{[2]} \cong (\mathbb{Z}/q)^I$. Indeed, for such G take S to be a free profinite group of the appropriate rank [NSW08, 3.5.4]. Conversely, a cover $\pi: S \rightarrow G$ induces an epimorphism $\pi(p): S(p) \rightarrow G(p)$ of the maximal pro- p quotients. Then $H^2(S(p)) = 0$ [CEM12, Lemma 6.5], so $S(p)$ is a free pro- p group [NSW08, Prop. 3.5.17]. Thus $G^{[2]} \cong S^{[2]} \cong S(p)^{[2]} \cong (\mathbb{Z}/q)^I$ for some I .

We call (T, T_0, α) a **cohomological duality triple** if for every profinite group G the following conditions hold:

- (A1) $T(G), T_0(G)$ are normal subgroups of G ;
- (A2) $T(G)^q[T(G), G] \leq T_0(G) \leq T(G) \leq G^{(2)}$;
- (A3) for every epimorphism $\pi: G \rightarrow \bar{G}$ one has $T(\bar{G}) = \pi(T(G))$ and $T_0(\bar{G}) = \pi(T_0(G))$;
- (A4) if there is a cover $S \rightarrow G$, then $A(G/T(G))$ is dual to $(T(G), T_0(G))$.

We list three basic examples of cohomological duality triples. Condition (A2) for example (2) was shown in Remark 2.1(2), and (A4) for all these

examples is just Examples 3.3. The verification of the remaining conditions is straightforward.

Examples 4.2. (1) Let $T(G) = G^{(2)}$, $T_0(G) = G_{(3)}$, and let α_G be the cup product on $H^1(G)^{\otimes 2}$ and the trivial map on $H^1(G)^{\otimes r}$ for $r \neq 2$. Thus $A(G) = H^2(G)_{\text{dec}}$.

(2) Let $T(G) = G^{(2)}$, $T_0(G) = G^{(3)}$, and let α_G be β_G on $H^1(G)$, the cup product on $H^1(G)^{\otimes 2}$, and the trivial map on $H^1(G)^{\otimes r}$ for $r \neq 1, 2$. Then $A(G) = \text{Im}(\beta_G) + H^2(G)_{\text{dec}}$. If there is a cover $S \rightarrow G$, then by Lemma 2.3 and Example 4.1, $A(G^{[2]}) = H^2(G^{[2]})$, so indeed, Example 3.3(2) applies.

(3) Let $T(G) = G^{(2)}$, $T_0(G) = G^{q^2}[G, G]$, and let $\alpha_G = \beta_G$ on $H^1(G)$ and $\alpha_G = 0$ on $H^1(G)^{\otimes r}$ for $r \neq 1$. Thus $A(G) = \text{Im}(\beta_G)$.

Remark 4.3. Let (T, T_0, α) be a cohomological duality triple, $\pi: G_1 \rightarrow G_2$ an epimorphism of profinite groups, and suppose that there are covers $S \rightarrow G_i$, $i = 1, 2$, which commute with π . Then π induces an isomorphism $G_1/T(G_1) \xrightarrow{\sim} G_2/T(G_2)$. As $T(G_i) \leq G_i^{(2)}$, the induced map $H^{\otimes*}(G_2) \rightarrow H^{\otimes*}(G_1)$ is therefore an isomorphism. Hence $\pi_2^*(A(G_2)) = A(G_1)$.

For a cohomological duality triple (T, T_0, α) and for $r \geq 0$, $t \geq 1$, let $C_{r,t}(G)$ be the \mathbb{Z}/q -submodule of $H^1(G)^{\otimes r}$ generated by all its elements $\psi_1 \otimes \cdots \otimes \psi_r$ such that $\alpha_G(\psi_{i_1} \otimes \cdots \otimes \psi_{i_t}) = 0$ for some $1 \leq i_1 < \cdots < i_t \leq r$. Thus $C_{r,t}(G) = 0$ for $r < t$. We define $H_{t,\alpha}^r(G) = H^1(G)^{\otimes r} / C_{r,t}(G)$ and a graded ring $H_{t,\alpha}^*(G) = \bigoplus_{r=0}^{\infty} H_{t,\alpha}^r(G)$. In particular, $H_{t,\alpha}^0(G) = \mathbb{Z}/q$. Since α is a natural transformation, so is $H_{t,\alpha}^*$. Note that α_G induces a homomorphism $\bar{\alpha}_G^t: H_{t,\alpha}^t(G) \rightarrow A(G)$.

Examples 4.4. (1) In Example 4.2(1), $H_{2,\alpha}^*(G) = \widehat{H^*(G)}$. The map $\bar{\alpha}_G^2: H_{2,\alpha}^2(G) \rightarrow A(G) = H^2(G)_{\text{dec}}$ is surjective, and is injective if and only if $H^*(G)$ is 2-quadratic.

(2) In Example 4.2(2), $H_{1,\alpha}^*(G) = H_{\text{Bock}}^*(G)$ and $H_{2,\alpha}^*(G) = \widehat{H^*(G)}$.

(3) In Example 4.2(3), $H_{1,\alpha}^*(G) = H_{\text{Bock}}^*(G)$ and the map $\bar{\alpha}_G^1: H_{1,\alpha}^1(G) = H^1(G) / \text{Ker}(\beta_G) \rightarrow A(G) = \text{Im}(\beta_G)$ is an isomorphism.

5. QUOTIENTS THAT DETERMINE COHOMOLOGY

This section is devoted to proving Theorem A. More generally, let (T, T_0, α) be a cohomological duality triple. We show that, under some mild assumptions, the quotient $G/T_0(G)$ determines the graded rings $H_{t,\alpha}^*(G)$, $t \geq 1$, and is in fact the minimal such quotient:

Theorem 5.1. *Assume that there is a cover $S \rightarrow G$. Let N be a normal subgroup of G contained in $T(G)$, and consider the following conditions:*

- (a) $N \leq T_0(G)$;
- (b) $\inf_G: A(G/N) \rightarrow A(G)$ is an isomorphism;
- (c) $\inf_G: H_{t,\alpha}^*(G/N) \rightarrow H_{t,\alpha}^*(G)$ is an isomorphism for every t .

Then (a) \Leftrightarrow (b) \Rightarrow (c). Moreover, if there exist $r \geq 1$ such that $\bar{\alpha}_G^r: H_{r,\alpha}^r(G) \rightarrow A(G)$ is injective and $\bar{\alpha}_{G/N}^r: H_{r,\alpha}^r(G/N) \rightarrow A(G/N)$ surjective, then (a)–(c) are equivalent.

Before we proceed with the proof of Theorem 5.1, we apply it to Examples 4.2, to deduce the following special cases.

Corollary 5.2. *Let G be a profinite group with $G^{[2]} \cong (\mathbb{Z}/q)^I$ for some I . Let N be a normal subgroup of G contained in $G^{(2)}$.*

- (1) *If $H^*(G)$ is 2-quadratic, then the following conditions are equivalent:*
 - (a) $N \leq G_{(3)}$;
 - (b) $\inf_G: H^2(G/N)_{\text{dec}} \rightarrow H^2(G)_{\text{dec}}$ is an isomorphism;
 - (c) $\inf_G: H^*(G/N)_{\text{dec}} \rightarrow H^*(G)_{\text{dec}}$ is an isomorphism.
- (2) *The following conditions are equivalent:*
 - (a) $N \leq G^{(3)}$;
 - (b) $\inf_G: \text{Im}(\beta_{G/N}) + H^2(G/N)_{\text{dec}} \rightarrow \text{Im}(\beta_G) + H^2(G)_{\text{dec}}$ is an isomorphism;
- (3) *The following conditions are equivalent:*
 - (a) $N \leq G^{q^2}[G, G]$;
 - (b) $\inf_G: \text{Im}(\beta_{G/N}) \rightarrow \text{Im}(\beta_G)$ is an isomorphism;
 - (c) $\inf_G: H^*(G/N)_{\text{Bock}} \rightarrow H^*(G)_{\text{Bock}}$ is an isomorphism.

Proof. Everything follows directly from Theorem 5.1, using Example 4.1 and Examples 4.4, except for the equivalence with (1)(c). Since 1(c) trivially implies 1(b), it suffices to show that 1(b) implies 1(c). Indeed, by Theorem 5.1 and 1(b), $\inf_G: H_{t,\alpha}^*(G/N) \rightarrow H_{t,\alpha}^*(G)$ is an isomorphism for every t . For $t = 2$ this means that $\inf_G: \widehat{H^*(G/N)} \rightarrow \widehat{H^*(G)}$ is bijective. The functoriality of ω gives a commutative square

$$\begin{array}{ccc} \widehat{H^*(G/N)} & \xrightarrow[\sim]{\inf_G} & \widehat{H^*(G)} \\ \omega_{G/N} \downarrow & & \wr \downarrow \omega_G \\ H^*(G/N)_{\text{dec}} & \xrightarrow{\inf_G} & H^*(G)_{\text{dec}}. \end{array}$$

Since ω_G is by assumption bijective, the lower inflation is bijective. \square

In view of Theorem 2.4, (1) implies Theorem A. Note that in (1) the equivalence (a) \Leftrightarrow (b) holds even without the 2-quadraticness assumption.

For the proof of Theorem 5.1 we first show:

Proposition 5.3. *Let $\pi_i: S \rightarrow G_i$, $i = 1, 2$, be covers and $\pi: G_1 \rightarrow G_2$ an epimorphism with $\pi \circ \pi_1 = \pi_2$. The following conditions are equivalent:*

- (a) *the induced map $G_1/T_0(G_1) \rightarrow G_2/T_0(G_2)$ is an isomorphism;*
- (b) *$\text{Ker}(\pi) \leq T_0(G_1)$;*
- (c) *the induced map $A(\pi): A(G_2) \rightarrow A(G_1)$ is an isomorphism;*
- (d) *the induced map $A(\pi): A(G_2) \rightarrow A(G_1)$ is a monomorphism.*

Proof. (a) \Rightarrow (b), (c) \Rightarrow (d): Trivial.

(b) \Rightarrow (a): By (A3), $T_0(G_2) = \pi(T_0(G_1))$. Hence the kernel of the induced epimorphism $G_1/T_0(G_1) \rightarrow G_2/T_0(G_2)$ is $\text{Ker}(\pi)T_0(G_1)/T_0(G_1)$, which is trivial by (b).

(d) \Rightarrow (a): The map $\bar{\pi}: G_1/T(G_1) \rightarrow G_2/T(G_2)$ induced by π is an isomorphism. Hence so is $\pi_1^*: H^1(G_2) \rightarrow H^1(G_1)$. By (A4), π induces a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & K(G_2) & \xrightarrow{\text{trg}} & A(G_2/T(G_2)) & \xrightarrow{\text{inf}} & A(G_2) \\ & & \downarrow & & \downarrow A(\bar{\pi}) \wr & & \downarrow A(\pi) \\ 0 & \longrightarrow & K(G_1) & \xrightarrow{\text{trg}} & A(G_1/T(G_1)) & \xrightarrow{\text{inf}} & A(G_1). \end{array}$$

By the assumptions and the snake lemma, the left vertical map is an isomorphism. Passing to duals using Proposition 3.1(a), we obtain that the induced map $T(G_1)/T_0(G_1) \rightarrow T(G_2)/T_0(G_2)$ is an isomorphism. Now apply the snake lemma for the commutative diagram with exact rows

$$\begin{array}{ccccccc} 1 & \longrightarrow & T(G_1)/T_0(G_1) & \longrightarrow & G_1/T_0(G_1) & \longrightarrow & G_1/T(G_1) \longrightarrow 1 \\ & & \wr \downarrow & & \downarrow & & \downarrow \wr \\ 1 & \longrightarrow & T(G_2)/T_0(G_2) & \longrightarrow & G_2/T_0(G_2) & \longrightarrow & G_2/T(G_2) \longrightarrow 1. \end{array}$$

(a) \Rightarrow (c): Consider the commutative diagram

$$\begin{array}{ccccc} A(G_2/T(G_2)) & \xrightarrow{\text{inf}} & A(G_2/T_0(G_2)) & \xrightarrow{\text{inf}} & A(G_2) \\ \downarrow & & \downarrow & & \downarrow A(\pi) \\ A(G_1/T(G_1)) & \xrightarrow{\text{inf}} & A(G_1/T_0(G_1)) & \xrightarrow{\text{inf}} & A(G_1), \end{array}$$

where the vertical maps are induced by π . (a) implies that the middle vertical map is an isomorphism. Since $T_0(G_i) \leq T(G_i) \leq G_i^{(2)}$, the inflations $H^1(G_i/T(G_i)) \rightarrow H^1(G_i/T_0(G_i)) \rightarrow H^1(G_i)$ are isomorphisms, $i = 1, 2$. Hence the horizontal inflation maps are surjective. By Proposition 3.4, in each row the kernel of the left inflation map equals the kernel of the composed inflation map. It follows that $A(\pi)$ is an isomorphism. \square

Proposition 5.4. *In the setup of Proposition 5.3, conditions (a)–(d) imply:*

(e) π induces for every $t \geq 1$ an isomorphism $\pi^*: H_{t,\alpha}^*(G_2) \rightarrow H_{t,\alpha}^*(G_1)$.

Moreover, if there exists $r \geq 1$ with $\bar{\alpha}_{G_1}^r: H_{r,\alpha}^r(G_1) \rightarrow A(G_1)$ injective and $\bar{\alpha}_{G_2}^r: H_{r,\alpha}^r(G_2) \rightarrow A(G_2)$ surjective, then (e) is equivalent to (a)–(d).

Proof. Since the induced map $G_1^{[2]} \rightarrow G_2^{[2]}$ is an isomorphism, so is $\pi_1^*: H^1(G_2) \rightarrow H^1(G_1)$, and we obtain a commutative square

$$\begin{array}{ccc} H^1(G_2)^{\otimes t} & \xrightarrow[\sim]{(\pi_1^*)^{\otimes t}} & H^1(G_1)^{\otimes t} \\ \alpha_{G_2} \downarrow & & \downarrow \alpha_{G_1} \\ A(G_2) & \xrightarrow{A(\pi)} & A(G_1). \end{array}$$

Assuming (d), $(\pi_1^*)^{\otimes r}$ maps $C_{r,t}(G_2)$ bijectively onto $C_{r,t}(G_1)$ for every $t \geq 1$, and therefore $\pi^*: H_{t,\alpha}^*(G_2) \rightarrow H_{t,\alpha}^*(G_1)$ is an isomorphism.

For the last assertion, assume (e) and consider the commutative square

$$\begin{array}{ccc} H_{r,\alpha}^r(G_2) & \xrightarrow[\sim]{\pi_r^*} & H_{r,\alpha}^r(G_1) \\ \bar{\alpha}_{G_2}^r \downarrow & & \downarrow \bar{\alpha}_{G_1}^r \\ A(G_2) & \xrightarrow{A(\pi)} & A(G_1). \end{array}$$

The assumptions imply that $A(\pi)$ is injective. \square

Consider the triple of Example 3.3(1) and let G_1 and G_2 be maximal pro- p Galois groups of fields containing a q th root of unity. Then the additional assumptions in Proposition 5.4 are satisfied for $r = 2$, by Theorem 2.4, Remark 2.6, and Remark 4.4(1). This proves the second equivalence of Theorem C.

Proof of Theorem 5.1. As $N \leq T(G)$, (A3) gives $T(G/N) = T(G)/N$. Therefore the composed map $S \rightarrow G \rightarrow G/N$ is a cover. Now apply Propositions 5.3 and 5.4 for the projection $\pi: G \rightarrow G/N$. \square

6. ISOMORPHISMS

We now apply the results of §5 to the case of pro- p groups.

Proposition 6.1. *Let (T, T_0, α) be a cohomological duality triple. Let $\pi_i: S \rightarrow G_i$, $i = 1, 2$, be covers and $\pi: G_1 \rightarrow G_2$ an epimorphism of pro- p groups with $\pi \circ \pi_1 = \pi_2$. Suppose that $A(G_2) = H^2(G_2)$. Then π is an isomorphism if and only if the induced map $G_1/T_0(G_1) \rightarrow G_2/T_0(G_2)$ is an isomorphism.*

Proof. The “only if” part follows from (A3). For the “if” part, recall that by [Ser65, Lemma 2], π is an isomorphism if and only if $\pi_r^*: H^r(G_2) \rightarrow H^r(G_1)$ is an isomorphism for $r = 1$ and a monomorphism for $r = 2$. As π_1^* commutes with the isomorphisms $H^1(G_i) \rightarrow H^1(S)$, $i = 1, 2$, it is also an isomorphism. Since the induced map $G_1/T_0(G_1) \rightarrow G_2/T_0(G_2)$ is an isomorphism and by Proposition 5.3, $\pi_2^*: A(G_2) = H^2(G_2) \rightarrow H^2(G_1)$ is a monomorphism. Hence π is an isomorphism. \square

We deduce the following strengthening of [CEM12, Remark 6.4, Th. D] (which deal with the quotients $G_i^{[3]}$):

Corollary 6.2. *Let $\pi: G_1 \rightarrow G_2$ be an epimorphism of pro- p groups inducing an isomorphism $\pi^{[2]}: G_1^{[2]} \xrightarrow{\sim} G_2^{[2]} \cong (\mathbb{Z}/q)^I$ for some I , and such that $H^2(G_2) = H^2(G_2)_{\text{dec}}$. Then π is an isomorphism if and only if the induced map $\pi_{[3]}: (G_1)_{[3]} \rightarrow (G_2)_{[3]}$ is an isomorphism.*

Proof. Choose bases (i.e., generating subsets converging to 1) \bar{Z}_i of $G_i^{[2]}$, $i = 1, 2$, such that $\pi^{[2]}(\bar{Z}_1) = \bar{Z}_2$. Lift \bar{Z}_1 to a subset Z_1 of G_1 , and let $Z_2 = \pi(Z_1)$. By the Frattini argument, Z_1, Z_2 generate G_1, G_2 , respectively. Let S be a free pro- p group with basis Z_1 . Let $\pi_1: S \rightarrow G_1$ be the natural epimorphism, let $\pi_2 = \pi \circ \pi_1$, and note that π_1, π_2 are covers. Now take the triple of Example 4.2(1) and apply Proposition 6.1. \square

In view of Remark 2.6, this implies the first equivalence of Theorem C.

Next Corollary 5.2(1) and Remark 2.6 give the following refinement of [CEM12, Prop. 9.1].

Corollary 6.3. *Let G_1, G_2 be profinite groups such that $(G_1)_{[3]} \cong (G_2)_{[3]}$ but $H^*(G_1) \not\cong H^*(G_2)$. Then at most one of G_1, G_2 can be isomorphic to the maximal pro- p Galois group $G_F(p)$ of a field F containing a root of unity of order q .*

As in [CEM12, §9], Corollary 6.3 can be used to show that various pro- p groups do not occur as $G_F(p)$ for F as above.

Examples 6.4. We assume that $q = p$ is prime.

(1) Let S be a free pro- p group and R a normal subgroup of S such that $R \leq S_{(3)}$ and $S \not\cong S/R$. Take in Corollary 6.3 $G_1 = S$ and $G_2 = S/R$. Then $(G_1)_{[3]} \cong (G_2)_{[3]}$ and $H^1(G_1) \cong H^1(G_2)$ (as $R \leq S^{(2)}$). Hence G_1, G_2 have the same rank [NSW08, Prop. 3.9.1]. Since a free pro- p group is determined by its rank, G_2 is not free pro- p . Therefore $H^2(S) = 0 \neq H^2(G_2)$ [NSW08, Cor. 3.9.5]. Now G_1 is realizable as an absolute Galois group of a field of characteristic $\neq p$ [FJ05, Cor. 23.1.2], and such a field automatically contains a p th root of unity. By Corollary 6.3 $G_2 \not\cong G_F(p)$ for any field F containing a p th root of unity.

(2) Take in (1) $G_1 = S = \mathbb{Z}_p$ and $R = (\mathbb{Z}_p)_{(3)} = \delta p \mathbb{Z}_p$ (with δ as in §2). Thus $G_2 = S/R = \mathbb{Z}/4$ for $p = 2$, and $G_2 = \mathbb{Z}/p$ for p odd. Consequently, $G_2 \not\cong G_F(p)$ for any field F containing a p th root of unity. We recover Becker's generalization of the classical Artin–Schreier theorem [Bec74]: the order of an element in $G_F(p)$ can be only 1, 2, or ∞ .

7. QUOTIENTS DETERMINED BY COHOMOLOGY

In this section we prove a partial converse of Theorem 5.1, saying that for a cohomological duality triple (T, T_0, α) , and under some mild assumptions, $G/T_0(G)$ is determined by α_G and $G/T(G)$ (Theorem 7.2). In particular, this will prove Theorem B.

First consider a cover $\pi: S \rightarrow G$. Let $R = \text{Ker}(\pi)$. Then $R \leq T(S)$. In view of Proposition 3.1(a), there is a commutative diagram of perfect (substitution) pairings

$$(7.1) \quad \begin{array}{ccc} R/R^q[R, S] & \times & H^1(R)^S \longrightarrow \mathbb{Z}/q \\ \downarrow \iota & & \uparrow \text{res}_R \quad \parallel \\ T(S)/T_0(S) & \times & K(S) \longrightarrow \mathbb{Z}/q. \end{array}$$

Also, there is a commutative diagram with exact rows

$$(7.2) \quad \begin{array}{ccccccc} 1 & \longrightarrow & R & \longrightarrow & S & \xrightarrow{\pi} & G \longrightarrow 1 \\ & & \downarrow & & \parallel & & \downarrow \\ 1 & \longrightarrow & T(S) & \longrightarrow & S & \longrightarrow & S/T(S) \longrightarrow 1. \end{array}$$

Since $R \leq T(S) \leq S^{(2)}$, the inflation maps $H^1(G) \rightarrow H^1(S)$, $H^1(S/T(S)) \rightarrow H^1(S)$ are isomorphisms. By the 5-term sequence and as $H^2(S) = 0$, the two transgression maps arising from (7.2) are therefore isomorphisms. By

the functoriality of transgression [EM11, (2.2)], they commute. We get a commutative diagram

$$(7.3) \quad \begin{array}{ccccc} K(S) & \hookrightarrow & H^1(T(S))^S & \xrightarrow{\text{res}} & H^1(R)^S \\ \text{trg} \downarrow \wr & & \text{trg} \downarrow \wr & & \wr \downarrow \text{trg} \\ A(S/T(S)) & \hookrightarrow & H^2(S/T(S)) & \xrightarrow{\text{inf}} & H^2(G) \end{array}$$

where the left isomorphism is by (A4). Let g be the composite map $K(S) \rightarrow H^2(G)$ arising from this diagram. Let $\text{Ker}(g)^\vee$ denote the annihilator of $\text{Ker}(g)$ in $T(S)/T_0(S)$ under the lower pairing of (7.1).

Lemma 7.1. $G/T_0(G) \cong (S/T_0(S))/\text{Ker}(g)^\vee$.

Proof. By (7.3), $\text{Ker}(g)$ is the kernel of $\text{res}_R: K(S) \rightarrow H^1(R)^S$. By (7.1), $\text{Ker}(g)^\vee \cong \text{Im}(\iota) = RT_0(S)/T_0(S)$. Using (A3) we see that this is the kernel of the induced epimorphism $S/T_0(S) \rightarrow G/T_0(G)$. \square

Given covers $S \rightarrow G_i$, $i = 1, 2$, we have isomorphisms $S^{[2]} \rightarrow G_i^{[2]}$. They induce isomorphisms $H^1(G_i) \cong H^1(S)$ and $H^{\otimes*}(G_1) \xrightarrow{\sim} H^{\otimes*}(G_2)$.

Theorem 7.2. *Let (T, T_0, α) be a cohomological duality triple. Let $\pi_i: S \rightarrow G_i$ be covers, $i = 1, 2$, and $\sigma: H^{\otimes*}(G_1) \rightarrow H^{\otimes*}(G_2)$ the induced isomorphism. Suppose that there is a monomorphism $\tau: H^2(G_1) \rightarrow H^2(G_2)$ with $\alpha_{G_2} \circ \sigma = \tau \circ \alpha_{G_1}$. Then there is an isomorphism $G_1/T_0(G_1) \cong G_2/T_0(G_2)$ compatible with π_i , $i = 1, 2$.*

Proof. For $i = 1, 2$ there is a commutative diagram

$$\begin{array}{ccccc} H^{\otimes*}(S/T(S)) & \xrightarrow{\sim} & H^{\otimes*}(G_i/T(G_i)) & \xrightarrow[\sim]{\text{inf}} & H^{\otimes*}(G_i) \\ \alpha_{S/T(S)} \downarrow & & \alpha_{G_i/T(G_i)} \downarrow & & \downarrow \alpha_{G_i} \\ A(S/T(S)) & \xrightarrow{\sim} & A(G_i/T(G_i)) & \xrightarrow{\text{inf}} & A(G_i). \end{array}$$

Define a homomorphism $g_i: K(S) \rightarrow H^2(G_i)$ as above.

Given $\gamma \in H^{\otimes*}(S/T(S))$ let $\hat{\gamma}_i$ be the corresponding element in $H^{\otimes*}(G_i)$. Then γ maps trivially to $A(G_i)$ if and only if $\alpha_{G_i}(\hat{\gamma}_i) = 0$. Our assumption implies that $\alpha_{G_1}(\hat{\gamma}_1) = 0$ if and only if $\alpha_{G_2}(\hat{\gamma}_2) = 0$. Consequently, the kernels of $\text{inf}: A(S/T(S)) \rightarrow A(G_i) \subseteq H^2(G_i)$, $i = 1, 2$, coincide. Their preimages in $K(S)$ under transgression are $\text{Ker}(g_i)$, $i = 1, 2$ (see (7.3)), which therefore also coincide. Now use Lemma 7.1. \square

Applying this to Examples 4.2 we obtain:

Corollary 7.3. Assume that $G^{[2]} \cong (\mathbb{Z}/q)^I$ for some I .

- (1) $G^{[2]}$ and $\cup: H^1(G) \times H^1(G) \rightarrow H^2(G)$ determine $G_{[3]} = G/G_{(3)}$.
- (2) $G^{[2]}$, β_G , and $\cup: H^1(G) \times H^1(G) \rightarrow H^2(G)$ determine $G^{[3]} = G/G^{(3)}$.
- (3) $G^{[2]}$ and β_G determine $G/G^{q^2}[G, G]$.

In view of Theorem 2.4, (1) implies Theorem B.

8. PRESENTATIONS

Let (T, T_0, α) be again a cohomological duality triple. We use the techniques of the previous section to characterize the surjectivity of α_G in terms of group presentations. Let again $\pi: S \rightarrow G$ be a cover and $R = \text{Ker}(\pi)$. Note that $R^q[R, S] \leq R \cap T_0(S)$, by (A2).

Theorem 8.1. *There is a natural duality between $(R \cap T_0(S))/R^q[R, S]$ and the cokernel of $\text{inf}_G: A(G/T(G)) \rightarrow H^2(G)$.*

Proof. The induced map $S/T(S) \rightarrow G/T(G)$ is an isomorphism. From (7.3) we obtain a commutative diagram

$$\begin{array}{ccc} K(S) & \xrightarrow{\text{res}_R} & H^1(R)^S \\ \text{trg} \downarrow \wr & & \wr \downarrow \text{trg} \\ A(S/T(S)) & \xrightarrow{\text{inf}} & H^2(G). \end{array}$$

The right transgression maps $\text{Coker}(\text{res}_R)$ isomorphically onto the cokernel of $\text{inf}_G: A(G/T(G)) \rightarrow H^2(G)$. By (7.1), $\text{Coker}(\text{res}_R)$ is dual to $\text{Ker}(\iota) = (R \cap T_0(S))/R^q[R, S]$. \square

Corollary 8.2. α_G is onto $H^2(G)$ if and only if $R^q[R, S] = R \cap T_0(S)$.

Proof. The surjectivity of α_G is equivalent to the surjectivity of the inflation $\text{inf}_G: A(G/T(G)) \rightarrow H^2(G)$. Now use Theorem 8.1. \square

Applying this for Examples 4.2 we deduce:

Examples 8.3. Assume that $G^{[2]} \cong (\mathbb{Z}/q)^I$ for some I .

- (1) $H^2(G) = H^2(G)_{\text{dec}}$ if and only if $R^q[R, S] = R \cap S_{(3)}$.
- (2) $H^2(G) = H^2(G)_{\text{dec}} + \text{Im}(\beta_G)$ if and only if $R^q[R, S] = R \cap S^{(3)}$ (compare also [CEM12, Th. 7.1]).
- (3) $H^2(G) = \text{Im}(\beta_G)$ if and only if $R^q[R, S] = R \cap S^{q^2}[S, S]$.

By Theorem 2.4, if $G = G_F$ is the absolute Galois group of a field F containing a root of unity of order q , then (1), and therefore (2), are valid.

9. $T_0(G)$ AS AN INTERSECTION

Let (T, T_0, α) be again a cohomological duality triple. In this section we present $T_0(G)$ as the intersection of all open normal subgroups M of the profinite group G with G/M contained in a (functorially given) list $\mathcal{L}(G)$ of finite groups. In particular, this will imply Theorem D.

Following [EM11], we say that G has **Galois relation type** if

- (i) $G^{[2]} \cong (\mathbb{Z}/q)^I$ for some set I ;
- (ii) there exists $\xi \in H^1(G)$ with $\beta_G(\psi) = \psi \cup \xi$ for every $\psi \in H^1(G)$;
- (iii) the kernel of $\inf: H^2(G^{[2]})_{\text{dec}} \rightarrow H^2(G)$ is generated by cup products $\psi \cup \psi'$, with $\psi, \psi' \in H^1(G^{[2]})$.

By Theorem 2.4 and Lemma 2.2, this holds when $G = G_F$ is the absolute Galois group of a field F containing a root of unity of order q (this was earlier shown in [EM11, Prop. 3.2]).

Definition 9.1. A **special set** for the profinite group G with respect to (T, T_0, α) will be a set Σ of pairs $(\bar{G}, \bar{\varphi})$ such that \bar{G} is a finite quotient of $G^{[2]}$, $\bar{\varphi} \in H^{\otimes*}(\bar{G})$, and the kernel of $\inf_G: A(G/T(G)) \rightarrow H^2(G)$ is generated by the elements $\alpha_{G/T(G)}(\inf_{G/T(G)}(\bar{\varphi}))$ with $(\bar{G}, \bar{\varphi}) \in \Sigma$.

Examples 9.2. Let G be a profinite group of Galois relation type.

(1) Consider the cohomological duality triple of Example 4.2(1). Take $\psi, \psi' \in H^1(G^{[2]})$ such that $\psi \cup \psi' = \alpha_{G^{[2]}}(\psi \otimes \psi') \neq 0$ is in the kernel of $\inf: H^2(G^{[2]})_{\text{dec}} \rightarrow H^2(G)$. Let $\bar{G} = G^{[2]}/(\text{Ker}(\psi) \cap \text{Ker}(\psi'))$ and take $\bar{\psi}, \bar{\psi}' \in H^1(\bar{G})$ with $\psi = \inf_{G^{[2]}}(\bar{\psi})$, $\psi' = \inf_{G^{[2]}}(\bar{\psi}')$. By (iii), the set of all such pairs $(\bar{G}, \bar{\psi} \otimes \bar{\psi}')$ is a special set for G .

(2) Consider the triple of Example 4.2(2). By (i) and Lemma 2.3,

$$H^2(G^{[2]}) = \text{Im}(\beta_{G^{[2]}}) + H^2(G^{[2]})_{\text{dec}} = A(G/T(G)).$$

We first claim that $K' = \text{Ker}(\inf_G: H^2(G^{[2]}) \rightarrow H^2(G))$ is generated by elements of the form $\alpha_{G^{[2]}}(-\psi \oplus (\psi \otimes \psi')) = -\beta_{G^{[2]}}(\psi) + \psi \cup \psi'$, where $\psi, \psi' \in H^1(G^{[2]})$, $\psi \neq 0$, and $-\psi \oplus (\psi \otimes \psi')$ is considered as an element of $H^{\otimes*}(G^{[2]})$ (compare [EM11, Prop. 4.3]). Indeed, for ξ as in (ii) we take $\xi_0 \in H^1(G^{[2]})$ with $\xi = \inf_G(\xi_0)$. Let $\theta = \beta_{G^{[2]}}(\eta) + \sum_{i=1}^n \psi_i \cup \psi'_i \in K'$, where $\eta, \psi_i, \psi'_i \in H^1(G^{[2]})$. Then also $\eta \cup \xi_0 + \sum_{i=1}^n \psi_i \cup \psi'_i$ is in K' , and by (iii),

it can be written as $\sum_{j=1}^m \chi_j \cup \chi'_j$, with $\chi_j, \chi'_j \in H^1(G^{[2]})$ and $\chi_j \cup \chi'_j \in K'$ for each j . Hence

$$\begin{aligned} \theta &= \beta_{G^{[2]}}(\eta) + (-\eta) \cup \xi_0 + \sum_{j=1}^m \chi_j \cup \chi'_j \\ &= (\beta_{G^{[2]}}(\eta) + (-\eta) \cup \xi_0) + \sum_{j=1}^m (-\beta_{G^{[2]}}(\chi_j) + \chi_j \cup (\chi'_j + \xi_0)) \\ &\quad + \sum_{j=1}^m (\beta_{G^{[2]}}(\chi_j) + (-\chi_j) \cup \xi_0) \end{aligned}$$

and this sum is in K' , proving the claim.

Now given $-\psi \oplus (\psi \otimes \psi')$ as above, let $\bar{G} = G^{[2]}/(\text{Ker}(\psi) \cap \text{Ker}(\psi'))$ and take $\bar{\psi}, \bar{\psi}' \in H^1(\bar{G})$ with $\psi = \inf_{G^{[2]}}(\bar{\psi})$, $\psi' = \inf_{G^{[2]}}(\bar{\psi}')$. The set Σ of all pairs $(\bar{G}, -\bar{\psi} + (\bar{\psi} \otimes \bar{\psi}'))$ is special for G .

(3) Consider the cohomological duality triple of Example 4.2(3). Trivially, the kernel of $\inf_G: \text{Im}(\beta_{G^{[2]}}) \rightarrow \text{Im}(\beta_G)$ is generated by elements $\beta_{G^{[2]}}(\psi)$, with $\psi \in H^1(G^{[2]})$. For such ψ let $\bar{G} = G^{[2]}/\text{Ker}(\psi)$ and take $\bar{\psi} \in H^1(\bar{G})$ with $\psi = \inf(\bar{\psi})$. The set Σ of all pairs $(\bar{G}, \bar{\psi})$ is special for G .

For the rest of this section we assume that $q = p$ is prime. When $p \neq 2$ let H_{p^3} be the Heisenberg group of order p^3 (see the Introduction), and let

$$M_{p^3} = \langle r, s \mid r^{p^2} = s^p = 1, r^p = [r, s] \rangle$$

be the unique nonabelian group of odd order p^3 and exponent p^2 . Let D_4 be the dihedral group of order 8.

Given a special set Σ for G with respect to (T, T_0, α) , we choose for every $(\bar{G}, \bar{\varphi}) \in \Sigma$ a central extension

$$(9.1) \quad \omega: \quad 0 \rightarrow \mathbb{Z}/p \rightarrow B \rightarrow \bar{G} \rightarrow 1$$

corresponding to $\alpha_{\bar{G}}(\bar{\varphi}) \in H^2(\bar{G})$. Note that B depends only on $\alpha_{\bar{G}}(\bar{\varphi})$. Let $\mathcal{L}(G)$ be the class of all (isomorphism classes of) finite groups B arising in this way.

Examples 9.3. Suppose that G has Galois relation type.

(1) Let (T, T_0, α) be as in Example 4.2(1) and let Σ be the special set for G as in Example 9.2(1). Consider $(\bar{G}, \bar{\psi} \otimes \bar{\psi}') \in \Sigma$. Thus $\bar{\psi}, \bar{\psi}' \in H^1(\bar{G})$, $\bar{\psi} \cup \bar{\psi}' \neq 0$, and $\text{Ker}(\bar{\psi}) \cap \text{Ker}(\bar{\psi}') = \{1\}$. Let

$$\omega: \quad 0 \rightarrow \mathbb{Z}/p \rightarrow B \rightarrow \bar{G} \rightarrow 1$$

be the central extension corresponding to $\bar{\psi} \cup \bar{\psi}'$.

When $p \neq 2$, $\bar{\psi}, \bar{\psi}'$ are \mathbb{F}_p -linearly independent, $\bar{G} \cong (\mathbb{Z}/p)^2$ and $B \cong H_{p^3}$ [EM11, Prop. 9.1(f)]. Hence $\mathcal{L}(G) = \{H_{p^3}\}$.

Next let $p = 2$. When $\bar{\psi} = \bar{\psi}'$ we have $\bar{G} \cong \mathbb{Z}/2$ and $B \cong \mathbb{Z}/4$ [EM11, Prop. 9.1(c)]. Otherwise $\bar{\psi}, \bar{\psi}'$ are \mathbb{F}_p -linearly independent, $\bar{G} \cong (\mathbb{Z}/2)^2$ and $B \cong D_4$ [EM11, Prop. 9.1(e)]. We conclude that $\mathcal{L}(G) = \{\mathbb{Z}/4, D_4\}$.

(2) Let (T, T_0, α) be as in Example 4.2(2) and let Σ be the special set for G as in Example 9.2(2). Consider $(\bar{G}, -\bar{\psi} \oplus (\bar{\psi} \otimes \bar{\psi}')) \in \Sigma$. Thus $\bar{\psi}, \bar{\psi}' \in H^1(\bar{G})$, $\bar{\psi} \neq 0$, and $\text{Ker}(\bar{\psi}) \cap \text{Ker}(\bar{\psi}') = \{1\}$. Let

$$\omega : \quad 0 \rightarrow \mathbb{Z}/p \rightarrow B \rightarrow \bar{G} \rightarrow 1$$

be the central extension corresponding to $-\beta_{\bar{G}}(\bar{\psi}) + \bar{\psi} \cup \bar{\psi}'$.

When $p \neq 2$ and $\bar{\psi}, \bar{\psi}'$ are \mathbb{F}_p -linearly independent, $\bar{G} \cong (\mathbb{Z}/p)^2$ and $B \cong M_{p^3}$ [EM11, Prop. 9.4]. When $p \neq 2$ and $\bar{\psi}, \bar{\psi}'$ are \mathbb{F}_p -linearly dependent, $\bar{G} \cong \mathbb{Z}/p$ and $B \cong \mathbb{Z}/p^2$ [EM11, Cor. 9.3].

(3) Let (T, T_0, α) be as in Example 4.2(3), and take Σ as in Example 9.2(3). By [EM11, Prop. 9.2], the central extension corresponding to $(\bar{G}, \bar{\psi}) \in \Sigma$ is

$$0 \rightarrow \mathbb{Z}/p \rightarrow \mathbb{Z}/p^2 \rightarrow \bar{G} (\cong \mathbb{Z}/p) \rightarrow 1,$$

so $\mathcal{L}(G) = \{\mathbb{Z}/p^2\}$.

Theorem 9.4. *Suppose that Σ is a special set for the profinite group G with respect to the cohomological duality triple (T, T_0, A) . Then*

$$T_0(G) = T(G) \cap \bigcap \{M \trianglelefteq G \mid G/M \in \mathcal{L}(G)\}.$$

Proof. Let $(\bar{G}, \bar{\varphi}) \in \Sigma$ and ω a central extension as in (9.1). Since \bar{G} is a quotient of $G^{[2]}$ it is also a quotient of $G/T(G)$. Let $\text{pr}: B \times_{\bar{G}} (G/T(G)) \rightarrow B$ be the projection from the fibred product. The central extension

$$\hat{\omega} : \quad 0 \rightarrow \mathbb{Z}/p \rightarrow B \times_{\bar{G}} (G/T(G)) \rightarrow G/T(G) \rightarrow 1$$

then corresponds to $\inf_{G/T(G)}(\alpha_{\bar{G}}(\bar{\varphi}))$ [EM11, Remark 6.1].

Next let A_0 be the set of all elements $\alpha_{G/T(G)}(\inf_{G/T(G)}(\bar{\varphi}))$, where $(\bar{G}, \bar{\varphi}) \in \Sigma$. Thus A_0 generates the kernel of $\inf: A(G/T(G)) \rightarrow H^2(G)$. Consider homomorphisms $\Psi: G \rightarrow B$, $\hat{\Psi}: G \rightarrow B \times_{\bar{G}} (G/T(G))$ as in the following diagram. For every Ψ making the lower triangle commutative there is a

unique $\hat{\Psi}$ making the upper triangle commutative with $\Psi = \text{pr} \circ \hat{\Psi}$.

$$\begin{array}{ccccccc}
 & & & & G & & \\
 & & & & \downarrow & & \\
 \hat{\omega} : & 0 \longrightarrow \mathbb{Z}/p \longrightarrow B \times_{\bar{G}} (G/T(G)) & \xrightarrow{\quad} & G/T(G) & \longrightarrow & 1 \\
 & \parallel & \downarrow \text{pr} & \swarrow \Psi & \downarrow & & \\
 \omega : & 0 \longrightarrow \mathbb{Z}/p \longrightarrow B & \xrightarrow{\quad \pi \quad} & \bar{G} & \longrightarrow & 1.
 \end{array}$$

Note that $\text{Ker}(\hat{\Psi}) = T(G) \cap \text{Ker}(\Psi)$. Furthermore, if π maps a proper subgroup B_0 of B onto \bar{G} , then $\pi|_{B_0} : B_0 \rightarrow \bar{G}$ is an isomorphism. Since ω is non-split, Ψ is therefore surjective. Now by condition (f) of Proposition 3.2, $T_0(G) = \bigcap \text{Ker}(\hat{\Psi}) = T(G) \cap \bigcap \text{Ker}(\Psi)$ for all Ψ as above ($T_0(G) = T(G)$ when there is no such Ψ). The kernels $\text{Ker}(\Psi)$ are just the normal subgroups M of G such that $G/M = B \in \mathcal{L}(G)$. \square

We now apply Theorem 9.4 to Examples 4.2 to derive concrete presentations of the canonical subgroups discussed so far as intersections.

Examples 9.5. Suppose that G has Galois relation type.

(1) Let (T, T_0, α) be as in Example 4.2(1) and let Σ be the special set for G as in Example 9.2(1). By Example 9.3(1), $\mathcal{L}(G) = \{H_{p^3}\}$ when $p \neq 2$ and $\mathcal{L}(G) = \{\mathbb{Z}/4, D_4\}$ when $p = 2$. In the first case we obtain from Theorem 9.4 that

$$\begin{aligned}
 G_{(3)} &= G^{(2)} \cap \bigcap \{M \trianglelefteq G \mid G/M \cong H_{p^3}\} \\
 &= \bigcap \{M \trianglelefteq G \mid G/M \cong 1, \mathbb{Z}/p, H_{p^3}\}.
 \end{aligned}$$

In view of Theorem 2.4, this gives Theorem D.

In the second case $G_{(3)} = G^{(3)}$ (Remark 2.1(1)), and we obtain that

$$G_{(3)} = \bigcap \{M \trianglelefteq G \mid G/M \cong 1, \mathbb{Z}/2, \mathbb{Z}/4, D_4\}.$$

This last fact was proved by Villegas [Vil88], and Mináč and Spira [MSp96, Cor. 2.18] when G is an absolute Galois group of a field, and in [EM11, Cor. 11.3 and Prop. 3.2] for general profinite groups of Galois relation type. Moreover, $\mathbb{Z}/2$ can be omitted from this list if $G \not\cong \mathbb{Z}/2$ [EM11, Cor. 11.4].

(2) Let (T, T_0, α) be as in Example 4.2(2) and let Σ be the special set for G as in Example 9.2(2). We may assume that $p \neq 2$, since otherwise $G^{(3)} = G_{(3)}$, and this subgroup was described in (1). Now by Example

9.3(2), $\mathcal{L}(G) = \{M_{p^3}, \mathbb{Z}/p^2\}$. By [EM11, Prop. 10.2], every epimorphism $G \rightarrow \mathbb{Z}/p$ breaks via \mathbb{Z}/p^2 or via M_{p^3} . Consequently, Theorem 9.4 gives

$$G^{(3)} = \bigcap \{M \trianglelefteq G \mid G/M \cong 1, \mathbb{Z}/p^2, M_{p^3}\}.$$

This result was earlier proved in [EM11].

(3) Let (T, T_0, α) be as in Example 4.2(3), and take Σ as in Example 9.2(3). By Example 9.3(2), $\mathcal{L}(G) = \{\mathbb{Z}/p^2\}$. We get the equality (already noted in Example 3.3(3))

$$G^{p^2}[G, G] = \bigcap \{M \trianglelefteq G \mid G/M \cong 1, \mathbb{Z}/p, \mathbb{Z}/p^2\}.$$

In fact, since a discrete abelian group of finite exponent is a direct sum of cyclic groups [Kap69, Th. 6], we get using Pontrjagin duality that

$$G^{p^n}[G, G] = \bigcap \{M \trianglelefteq G \mid G/M \cong \mathbb{Z}/p^j, j = 0, 1, \dots, n\}.$$

10. DUALITY IN FREE PRO- p GROUPS

In this section we prove the duality mentioned in Example 3.3(1). First we study the pairing in Proposition 3.1(b) when $G = S$ satisfies $H^2(S) = 0$ and when $T = S^{(2)}$ and $T_0 = S^{(3)}$. Given $\sigma \in S$, we write $\bar{\sigma}$ for its image in $S^{[2]}$. The next two lemmas extend computations in [Lab66, §2.3], [Koc02, §7.8] and [NSW08, 3.9.13]. Let $\delta = 1, 2$ be as in (2.1).

Lemma 10.1. *Let $\chi, \chi' \in H^1(S^{[2]})$ and $\sigma, \sigma' \in S$.*

- (a) *If there is a homomorphism $h: S \rightarrow \mathbb{Z}/q$ with $h(\sigma) = \bar{1}$, then $\langle \sigma^q S^{(3)}, \chi \cup \chi' \rangle = (q/\delta) \chi(\bar{\sigma}) \chi'(\bar{\sigma})$ ($= 0$ for q odd).*
- (b) *$\langle [\sigma, \sigma'] S^{(3)}, \chi \cup \chi' \rangle = \chi(\bar{\sigma}') \chi'(\bar{\sigma}) - \chi(\bar{\sigma}) \chi'(\bar{\sigma}')$.*

Proof. The cohomology class $\inf_S(\chi \cup \chi')$ in $H^2(S)$ is represented by the 2-cocycle $c(\sigma, \tau) = \chi(\bar{\sigma}) \chi'(\bar{\tau})$. Since $H^2(S) = 0$, there exists an inhomogeneous 1-cochain $u: S \rightarrow \mathbb{Z}/q$ with $\partial u = c$. Thus

$$(10.1) \quad \chi(\bar{\sigma}) \chi'(\bar{\tau}) = u(\sigma) + u(\tau) - u(\sigma\tau)$$

for all $\sigma, \tau \in S$. In particular, for $\sigma \in S$ and $\tau \in S^{(2)}$ we have $u(\sigma\tau) = u(\sigma) + u(\tau) = u(\tau\sigma)$. It follows that for $\tau \in S^{(2)}$ one has

$$\begin{aligned} u(\tau) &= u(\sigma^{-1}) + u(\sigma\tau) - \chi(\bar{\sigma}^{-1}) \chi'(\bar{\sigma}) \\ &= u(\sigma^{-1}) + u(\tau\sigma) - \chi(\bar{\sigma}^{-1}) \chi'(\bar{\sigma}) = u(\sigma^{-1}\tau\sigma). \end{aligned}$$

Thus the restriction v of u to $S^{(2)}$ belongs to $H^1(S^{(2)})^S$. By the definition of the transgression [NSW08, Prop. 1.6.6], $\text{trg}_{S^{[2]}}(v) = \chi \cup \chi'$. Consequently,

for every $\rho \in S^{(2)}$ we have

$$\langle \rho S^{(3)}, \chi \cup \chi' \rangle = \langle \rho S^{(3)}, \text{trg}_{S^{[2]}}(v) \rangle = -v(\rho) = -u(\rho).$$

(a) When $\sigma \in S^{(2)}$ both sides are zero. So assume that $\sigma \notin S^{(2)}$. Our assumption gives a homomorphism $h: S \rightarrow \mathbb{Z}/q$ with $h(\sigma) = u(\sigma)$. As $\partial h = 0$, we may replace u by $u - h$ to assume that $u(\sigma) = 0$. Using (10.1) we obtain inductively that $u(\sigma^i) = -\binom{i}{2} \chi(\bar{\sigma}) \chi'(\bar{\sigma})$. It remains to observe that $\binom{q}{2} \equiv q/\delta \pmod{q}$.

(b) Apply (10.1) with $\tau = 1$ to obtain $u(1) = 0$. Apply it with $\tau = \sigma^{-1}$ to further obtain $u(\sigma^{-1}) + u(\sigma) = -\chi(\bar{\sigma}) \chi'(\bar{\sigma})$. This gives

$$\begin{aligned} u((\sigma' \sigma)^{-1}) + u(\sigma' \sigma) &= -\chi(\bar{\sigma}' \bar{\sigma}) \chi'(\bar{\sigma}' \bar{\sigma}) \\ u(\sigma \sigma') &= u(\sigma) + u(\sigma') - \chi(\bar{\sigma}) \chi'(\bar{\sigma}') \\ -u(\sigma' \sigma) &= -u(\sigma') - u(\sigma) + \chi(\bar{\sigma}') \chi'(\bar{\sigma}) \end{aligned}$$

Adding these equalities we obtain

$$u((\sigma' \sigma)^{-1}) + u(\sigma \sigma') = -\chi(\bar{\sigma}' \bar{\sigma}) \chi'(\bar{\sigma}' \bar{\sigma}) + \chi(\bar{\sigma}') \chi'(\bar{\sigma}) - \chi(\bar{\sigma}) \chi'(\bar{\sigma}').$$

Hence, by (10.1) again,

$$\begin{aligned} u([\sigma, \sigma']) &= u((\sigma' \sigma)^{-1}) + u(\sigma \sigma') - \chi((\bar{\sigma} \bar{\sigma}')^{-1}) \chi'(\bar{\sigma} \bar{\sigma}') \\ &= \chi(\bar{\sigma}') \chi'(\bar{\sigma}) - \chi(\bar{\sigma}) \chi'(\bar{\sigma}'). \end{aligned}$$

as desired. \square

Lemma 10.2. *Let $\chi \in H^1(S^{[2]})$ and $\sigma, \sigma' \in S$.*

- (a) *If $\chi(\bar{\sigma}) = \bar{0}$, then $\langle \sigma^q S^{(3)}, \beta_{S^{[2]}}(\chi) \rangle = \bar{0}$.*
- (b) *If $\chi(\bar{\sigma}) = \bar{1}$, then $\langle \sigma^q S^{(3)}, \beta_{S^{[2]}}(\chi) \rangle = \bar{1}$.*
- (c) *$\langle [\sigma, \sigma'] S^{(3)}, \beta_{S^{[2]}}(\chi) \rangle = \bar{0}$.*

Proof. Define a section ι of the projection $\mathbb{Z}/q^2 \rightarrow \mathbb{Z}/q$ by $\iota(i + q\mathbb{Z}) = i + \mathbb{Z}/q^2$ for $0 \leq i \leq q - 1$. Let $\tilde{\chi} = \iota \circ \chi \in \text{Hom}(S^{[2]}, \mathbb{Z}/q^2)$. Then the cohomology class of $\beta_{S^{[2]}}(\chi)$ in $H^2(S^{[2]})$ is represented by the 2-cocycle

$$c(\bar{\sigma}, \bar{\tau}) = \frac{1}{q} (\tilde{\chi}(\bar{\sigma}) + \tilde{\chi}(\bar{\tau}) - \tilde{\chi}(\bar{\sigma} \bar{\tau})).$$

Inflating to $H^2(S) = 0$, we obtain an inhomogenous 1-cochain $u: S \rightarrow \mathbb{Z}/q$ with $\partial u = c$. Thus

$$(10.2) \quad u(\sigma) + u(\tau) - u(\sigma\tau) = \frac{1}{q} (\tilde{\chi}(\bar{\sigma}) + \tilde{\chi}(\bar{\tau}) - \tilde{\chi}(\bar{\sigma}\bar{\tau}))$$

for all $\sigma, \tau \in S$. Since $S^{[2]}$ is abelian, this implies that $u(\sigma\tau) = u(\tau\sigma)$, whence $u(\sigma^{-1}\tau\sigma) = u(\tau)$, for all $\sigma, \tau \in S$. It follows that the restriction v

of u to $S^{(2)}$ belongs to $H^1(S^{(2)})^S$. By the definition of the transgression, $\text{trg}_{S^{[2]}}(v)$ is represented by the 2-cocycle $\partial u = c$. Hence $\beta_{S^{[2]}}(\chi) = \text{trg}_{S^{[2]}}(v)$. Consequently, for every $\rho \in S^{(2)}$ we have

$$\langle \rho S^{(3)}, \beta_{S^{[2]}}(\chi) \rangle = \langle \rho S^{(3)}, \text{trg}_{S^{[2]}}(v) \rangle = -v(\rho) = -u(\rho).$$

(a) If $\chi(\sigma) = \bar{0}$, then $\tilde{\chi}(\sigma^i) = 0$ for every $i \geq 0$. It follows inductively from (10.2) that $u(\sigma^i) = iu(\sigma)$ for $i \geq 0$. Thus $u(\sigma^q) = \bar{0}$, as required.

(b) If $\chi(\sigma) = \bar{1}$, then $\tilde{\chi}(\sigma^i) = \bar{i}$ for $0 \leq i \leq q-1$, while $\tilde{\chi}(\sigma^q) = \bar{0}$. It follows from (10.2) by induction that $u(\sigma^i) = iu(\sigma)$ for $0 \leq i \leq q-1$, and $u(\sigma^q) = qu(\sigma) - \bar{1} = -\bar{1}$.

(c) As $(\sigma, \tau) \mapsto u(\sigma\tau)$ is symmetric, $u([\sigma, \sigma']) = \bar{0}$ for all $\sigma, \sigma' \in S$. \square

Next we also assume that $S^{[2]} \cong (\mathbb{Z}/q)^I$ for some index set I , e.g., S is a free profinite (or pro- p group) group. Fix a linear order $<$ on I . Choose a basis $\bar{\sigma}_i$, $i \in I$, of $S^{[2]}$, and lift it to elements σ_i , $i \in I$, of S . Let χ_i , $i \in I$, be the \mathbb{Z}/q -basis of $H^1(S^{[2]})$ dual to $\sigma_i S^{(2)}$, $i \in I$.

Proposition 10.3. (a) *Every $\sigma \in S^{(2)}$ can be uniquely written as*

$$\sigma = \prod_i \sigma_i^{a_i q} \prod_{i < j} [\sigma_i, \sigma_j]^{b_{ij}} \pmod{S^{(3)}}$$

for some $a_i, b_{ij} \in \mathbb{Z}/q$.

(b) *The lists*

(i) $\sigma_i^q S^{(3)}$, $i \in I$, and $[\sigma_i, \sigma_j] S^{(3)}$, $i, j \in I$, $i < j$;

(ii) $\beta_{S^{[2]}}(\chi_i)$, $i \in I$, and $\chi_i \cup \chi_j$, $i, j \in I$, $i < j$;

form dual bases of $S^{(2)}/S^{(3)}$ and $H^2(S^{[2]})$, respectively, with respect to $\langle \cdot, \cdot \rangle$.

Proof. (a) Use [NSW08, Prop. 3.9.13(i)] and a standard limit argument.

(b) By the definition of $S^{(2)}$, the list (i) generates $S^{(2)}/S^{(3)}$. By Lemma 10.1 and Lemma 10.2, the two lists are dual. Hence they form bases. \square

Proposition 10.4. *For $\chi \in H^1(S^{[2]})$ of order q one has $\chi \cup \chi = (q/\delta)\beta_{S^{[2]}}(\chi)$ ($= 0$ if $p > 2$).*

Proof. We may assume that $\chi = \chi_k$ for some $k \in I$. For $i \in I$ Lemma 10.1(a) and Lemma 10.2(a)(b) give

$$\langle \sigma_i^q S^{(3)}, \chi_k \cup \chi_k \rangle = (q/\delta)\chi_k(\sigma_i) = \langle \sigma_i^q S^{(3)}, (q/\delta)\beta_{S^{[2]}}(\chi_k) \rangle.$$

For $i, j \in I$, $i < j$, Lemma 10.1(b) and Lemma 10.2(c) give

$$\langle [\sigma_i, \sigma_j] S^{(3)}, \chi_k \cup \chi_k \rangle = 0 = \langle [\sigma_i, \sigma_j] S^{(3)}, (q/\delta)\beta_{S^{[2]}}(\chi_k) \rangle.$$

The assertion now follows by duality from Proposition 10.3(b). \square

From this and Proposition 10.3 we deduce

Corollary 10.5. *When $p > 2$ (resp., $p = 2$), the elements $\chi_i \cup \chi_j$, $i < j$ (resp. $i \leq j$) form a basis of $H^2(S^{[2]})_{\text{dec}}$.*

To this end let S be again a profinite group such that $H^2(S) = 0$ and $S^{[2]} \cong (\mathbb{Z}/q)^I$. Let σ_i, χ_i , $i \in I$, be bases as in the previous section. Write $S^{[2]} = \prod_{i \in I} C_i$, where $C_i = \langle \sigma_i S^{(2)} \rangle \cong \mathbb{Z}/q$. Then $\text{res}_{C_i}(\chi_i)$ generates $H^1(C_i) \cong \mathbb{Z}/q$, and $\beta_{C_i}(\text{res}_{C_i}(\chi_i))$ generates $H^2(C_i) \cong \mathbb{Z}/q$, e.g. by Corollary 10.5.

Lemma 10.6. *Let $\alpha \in H^2(S^{[2]})$ and let d_i , $i \in I$, and d_{ij} , $i < j$, $i, j \in I$, be the unique elements of \mathbb{Z}/q such that*

$$(10.3) \quad \alpha = \sum_i d_i \beta_{S^{[2]}}(\chi_i) + \sum_{i < j} d_{ij} \cdot \chi_i \cup \chi_j.$$

Then for every $k \in I$, one has $\text{res}_{C_k}(\alpha) = d_k \beta_{C_k}(\text{res}_{C_k}(\chi_k))$.

Proof. One has $\text{res}_{C_k}(\chi_i) = 0$ for $i \neq k$. Therefore

$$\text{res}_{C_k}(\beta_{S^{[2]}}(\chi_i)) = \beta_{C_k}(\text{res}_{C_k}(\chi_i)) = 0$$

for $i \neq k$, as well as

$$\text{res}_{C_k}(\chi_i \cup \chi_j) = \text{res}_{C_k}(\chi_i) \cup \text{res}_{C_k}(\chi_j) = 0$$

for all $i < j$, whence the desired equality. \square

We deduce the following local-global principle for groups of the form $(\mathbb{Z}/q)^I$.

Corollary 10.7. *There is an exact sequence*

$$0 \rightarrow H^2(S^{[2]})_{\text{dec}} \hookrightarrow H^2(S^{[2]}) \xrightarrow{\prod \delta_{\text{res}_C}} \prod_C H^2(C),$$

where C ranges over all cyclic subgroups of $S^{[2]}$ of order q .

Proof. Let $\alpha \in H^2(S^{[2]})$ and express it as in (10.3). By Corollary 10.5, $\alpha \in H^2(S^{[2]})_{\text{dec}}$ if and only if $\delta d_i \equiv 0 \pmod{q}$ for every $i \in I$. Since $\beta_{C_i}(\text{res}_{C_i}(\chi_i))$ generates $H^2(C_i)$, Lemma 10.6 shows that this is equivalent to $\delta \text{res}_{C_i}(\alpha) = 0$. It remains to note that every cyclic subgroup C of $S^{[2]}$ of order q occurs as $C_k = \langle \sigma_k S^{(2)} \rangle$ for some choice of σ_i, ψ_i . \square

Now let G be a profinite group. Given a subgroup C of $G^{[2]}$, take a normal subgroup M of G containing $G^{(2)}$ with $C = M/G^{(2)}$. Then $H^1(G^{(2)})^G \leq H^1(G^{(2)})^M$, and by the functoriality of the transgression, there is a commutative square

$$(10.4) \quad \begin{array}{ccc} H^1(G^{(2)})^G & \xrightarrow{\text{trg}_{G^{[2]}}} & H^2(G^{[2]}) \\ \downarrow & & \downarrow \text{res}_C \\ H^1(G^{(2)})^M & \xrightarrow{\text{trg}_C} & H^2(C). \end{array}$$

This makes condition (c) of the following proposition meaningful:

Proposition 10.8. *Let G be a profinite group with $G^{[2]} \cong (\mathbb{Z}/q)^I$ for some index set I . The following conditions on $\psi \in H^1(G^{(2)})^G$ are equivalent:*

- (a) $\text{trg}_{G^{[2]}}(\psi) \in H^2(G^{[2]})_{\text{dec}}$;
- (b) $\delta \text{trg}_C(\psi) = 0$ for every cyclic subgroup C of $G^{[2]}$ of order q ;
- (c) for every normal subgroup M of G containing $G^{(2)}$ with $M/G^{(2)} \cong \mathbb{Z}/q$ there exists $\hat{\psi} \in H^1(M)$ with $\delta\psi = \text{res}_{G^{(2)}}(\hat{\psi})$;
- (d) $\psi(G_{(3)}) = 0$.

Proof. (a) \Leftrightarrow (b): Since $G^{[2]} \cong S^{[2]}$ for some free pro- p group S , this follows from Corollary 10.7 and (10.4).

(b) \Leftrightarrow (c): Use the 5-term sequence associated with $C = M/G^{(2)}$.

(c) \Leftrightarrow (d): We may assume that $G \neq \{1\}$. Then G is the union of its subgroups M such that $G^{(2)} \leq M$ and $M/G^{(2)} \cong \mathbb{Z}/q$. Then $M^q \leq G^{(2)}$ and there is a split extension

$$1 \rightarrow G^{(2)}/M^q \rightarrow M/M^q \rightarrow \mathbb{Z}/q \rightarrow 0.$$

If there is $\hat{\psi} \in H^1(M)$ with $\delta\psi = \text{res}_{G^{(2)}}(\hat{\psi})$, then $(\delta\psi)(M^q) = \hat{\psi}(M^q) = \{0\}$. Conversely, if $\delta\psi$ vanishes on M^q , then it induces a homomorphism $\overline{\delta\psi} \in H^1(G^{(2)}/M^q)^M$. Since the above extension splits, $\overline{\delta\psi}$ extends to a homomorphism $M/M^q \rightarrow \mathbb{Z}/q$, so there is $\hat{\psi} \in H^1(M)$ as above.

Consequently, (c) is equivalent to $\psi(G^{\delta q}) = \{1\}$. But as $\psi \in H^1(G^{(2)})^G$, one always has $\psi([G^{(2)}, G]) = \{1\}$, so $\psi(G_{(3)}) = \psi(G^{\delta q})$. \square

The equivalence of (a) and (d) implies

Corollary 10.9. *Let G be a profinite group with $G^{[2]} \cong (\mathbb{Z}/q)^I$ for some index set I . Then $H^2(G^{[2]})_{\text{dec}}$ is dual to $(G^{(2)}, G_{(3)})$.*

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